MATH 590: QUIZ 12

Name:

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Follow the steps below to find the Jordan Canonical Form of A and the change of basis matrix P. You may use the fact that $p_A(x) = (x-1)^3$.

(i) Calculate E_1.
(ii) Write down the JCF of A, based upon your answer in (i) and $p_A(x)$.
(iii) Calculate $(A - 1 \cdot I)^2$.
(iv) Find v_3 not in the null space of $(A - 1 \cdot I)^2$.
(v) Take $v_2 := (A - 1 \cdot I)v_3$ and $v_1 := (A - 1 \cdot I)v_2$.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3, verify that $P^{-1}AP$ is the JCF of A.

Solution. (i) E_1 is the null space of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \stackrel{\text{EROs}}{\to} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for E_1.

(ii) The JCF of A is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, since the Jordan box associated to 1 is 3×3 and the number fo Jordan blocks equals one, the dimension of E_1.

(iii) $(A - I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}$.

(iv) The null space of $(A - I)^2$ is the null space of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so we can take $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(v) We have $v_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(vi) We have $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Thus,

$$
\begin{aligned}
P^{-1}AP &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
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